

$\Gamma: \forall z \in \mathbb{C} : \arg(z^n) = n \arg(z) + 2k\pi, k \in \mathbb{Z}$

D-d:

Lemma 1. $\forall z_1, z_2 \in \mathbb{C}$
(1) $\arg(z_1 \cdot z_2) = \arg(z_1) + \arg(z_2) + 2k\pi := \varphi$
(2) $|z_1 \cdot z_2| = |z_1| \cdot |z_2| \cdot (\cos(\arg(z_1) + i \sin(\arg(z_1)))$
D-d: Niech $\begin{cases} z_1 = |z_1| \cdot (\cos \varphi_1 + i \sin \varphi_1), & (\varphi_1, \varphi_2 \in \mathbb{R}) \\ z_2 = |z_2| \cdot (\cos \varphi_2 + i \sin \varphi_2) \end{cases}$

Zatem $\arg z_1 = \varphi_1, \arg z_2 = \varphi_2$

$$z_1 \cdot z_2 = |z_1| |z_2| (\cos \varphi_1 + i \sin \varphi_1) (\cos \varphi_2 + i \sin \varphi_2)$$

$$\begin{aligned} &= |z_1| |z_2| \left[\cos(\varphi_1 \cos \varphi_2 - \sin \varphi_1 \sin \varphi_2 \right. \\ &\quad \left. + i(\sin \varphi_1 \cos \varphi_2 + \cos \varphi_1 \sin \varphi_2)) \right] \\ &= |z_1| |z_2| \left(\cos(\varphi_1 + \varphi_2) + i \sin(\varphi_1 + \varphi_2) \right) \end{aligned}$$

Zatem $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) + 2k\pi, k \in \mathbb{Z}$.

Lemma 2. $\forall z_1, z_2, \dots, z_n \in \mathbb{C} :$

$$(1) \quad \prod_{i=1}^n z_i = \left(\prod_{i=1}^n |z_i| \right) \cdot \left(\cos(\varphi_1 + \dots + \varphi_n) + i \sin(\varphi_1 + \dots + \varphi_n) \right)$$

$$(2) \quad \arg \left(\prod_{i=1}^n z_i \right) = \sum_{i=1}^n \arg(z_i) + 2k\pi, k \in \mathbb{Z}$$

Dowód indukcyjny z Lemma 1.

Wniosek: (1) $\forall z \in \mathbb{C}: (1) z^n = |z|^n (\cos(n\varphi) + i \sin(n\varphi))$
(zbiór de Moivre'a)

Dowód z lemma 2.

$$\begin{aligned} & \text{Dla } n \in \mathbb{N}, \text{ skoro } z = r(\cos \varphi + i \sin \varphi), \text{ to } \\ & z^n = r^n (\cos(n\varphi) + i \sin(n\varphi)) \end{aligned}$$

$$\begin{aligned} & \text{Dla } n \in \mathbb{Z}, \text{ skoro } z = r(\cos \varphi + i \sin \varphi), \text{ to } \\ & z^n = r^n (\cos(n\varphi) + i \sin(n\varphi)) \end{aligned}$$

$$z^n = r^n (\cos(n\varphi) + i \sin(n\varphi))$$

$$\begin{aligned} & \text{Dla } n \in \mathbb{Q}, \text{ skoro } z = r(\cos \varphi + i \sin \varphi), \text{ to } \\ & z^n = r^n (\cos(n\varphi) + i \sin(n\varphi)) \end{aligned}$$

$$\begin{aligned} & \text{Dla } n \in \mathbb{R}, \text{ skoro } z = r(\cos \varphi + i \sin \varphi), \text{ to } \\ & z^n = r^n (\cos(n\varphi) + i \sin(n\varphi)) \end{aligned}$$

$$\begin{aligned} & \text{Dla } n \in \mathbb{C}, \text{ skoro } z = r(\cos \varphi + i \sin \varphi), \text{ to } \\ & z^n = r^n (\cos(n\varphi) + i \sin(n\varphi)) \end{aligned}$$

$$\begin{aligned} & \text{Dla } n \in \mathbb{C}, \text{ skoro } z = r(\cos \varphi + i \sin \varphi), \text{ to } \\ & z^n = r^n (\cos(n\varphi) + i \sin(n\varphi)) \end{aligned}$$